Nonlinear Transient Neutralization Theory of Ion Beams with Dissipation

H.E. Wilhelm*
Colorado State University, Fort Collins, Col.

An analytical theory of nonlinear neutralization waves generated by injection of electrons from a grid in the direction of a homogeneous ion beam of uniform velocity and infinite extension is presented. The electrons are assumed to interact with the ions through the self-consistent space charge field and by strong collective interactions. The associated nonlinear boundary-value problem is solved in closed form by means of a von Mises transformation. It is shown that the electron gas moves into the ion space in the form of a discontinuous neutralization wave. This periodic wave structure is damped out by intercomponent momentum transfer, i.e., after a few relaxation lengths a quasineutral beam results. The relaxation scale in space agrees with neutralization experiments of rarefied ion beams, if the collective momentum transfer between the electron and ion streams is assumed to be of the Buneman type.

I. Problem

THE neutralization of a rarefied ion beam by injection of an electron gas is a nonlinear process in which electron-ion interactions through the self-consistent space charge field play a predominant role. In the final stages of the neutralization, intercomponent momentum transfer and, to some extend, also diffusion resulting from pressure gradients become significant as dispersion mechanisms of periodic, local neutralization unbalances. As will be shown, the electron gas moves into a collisionless ion beam in the form of a regular nonlinear wave or a shock wave, depending on whether the electron injection density is above or below a critical value. For this reason, it is distinguished between a) "nonlinear neutralization waves" (no electron sheet crossing)[1,2] and b) "neutralization shock waves" (with electron sheet crossing).[1,2] Both phenomena exhibit a discontinuous wave front ahead of which the electron density is zero.

As a model, a quasihomogeneous ion beam of infinite extension is considered, i.e., the density N and velocity V of the beam are assumed to be uniform. The electrons are injected with a prescribed current density $j(t)$ in the downstream direction ($j \| V$). The homogeneous ion beam model is adequate for analyzing and understanding the basic properties of transient neutralization waves. The previous analytical work has been concerned with steady-state electron-ion neutralization without dissipation in homogeneous ion beams.[3-6]

II. Theoretical Formulation

An ion gas of homogeneous density N and uniform velocity V parallel to the x-axis is considered in the space $|x| \leq \infty$, $|y| \leq \infty$, $|z| \leq \infty$. This infinite ion beam is bisected by permeable grid in the plane $x=0$, from which electrons are ejected with the current density $j(t)$ in the direction of V (Fig. 1). The field equations for the electron velocity $v(x,t)$, electron density $n(x,t)$, and the self-consistent electric field $E(x,t)$ are[7]

$$\partial v/\partial t + v(\partial v/\partial x) = -(e/m)E - \nu(v-V) \tag{1}$$

$$\partial n/\partial t = -(\partial/\partial x)(nv) \tag{2}$$

$$\partial E/\partial x = 4\pi e(N-n) \tag{3}$$

where $e>0$ is the elementary charge and m is the electron mass. The term $-\nu(v-V)$ describes the intercomponent momentum transfer between electrons and ions (ν is the relaxation frequency). The injection of electrons with the current density $j(t)$ into the ion beam from the grid plane at $x=0$ is taken into consideration by the boundary conditions

$$\{(4\pi)^{-1}[\partial E(x,t)/\partial t] - en(x,t)v(x,t)\}_{x=0} = j(t) \tag{4}$$

$$v(x,t)_{x=0} = v_0 \quad t \geq 0 \tag{5}$$

$$E(x,t)_{x=0} = 0 \quad t \geq 0 \tag{6}$$

Equation (4) expresses the continuity of displacement and convection currents at the injection plane $x=+0$. Equations (5) and (6) represent the conditions for space charge limitation at the emitter.[3,4,8]

The electrons are assumed to be injected from the emitter at $x=0$ in the form of a step impulse. For this injection model, the electron current density in Eq. (4) is given by

$$j(t) = i_0 H(t) \tag{7}$$

where

$$H(t) = 1 \quad t \geq +0$$

$$H(t) = 0 \quad t \leq -0$$

and

$$i_0 = -en_0 v_0 < 0 \quad \sigma \equiv -Ne/i_0 > 0$$

The following mathematical considerations are valid for any dissipative mechanism characterized by a relaxation frequency ν, provided that the intercomponent friction force is linear in $(v-V)$. In the application to ion beam neutralization, Buneman's relaxation frequency,[1]

$$\nu = (m/M)^{1/3}(\omega/2\pi) \quad \omega = (4\pi N e^2/m)^{1/2} \tag{8}$$

is used, which takes into consideration the strong collective ("turbulence" due to the two-stream instability) momentum exchange between the electron and ion streams.

Equations (1-6) represent the nonlinear boundary-value problem for the transient neutralization process in the ion beam which takes place in the region $x>0$. Thermal effects are not taken into consideration, $\nabla p = 0$. The neglect of ther-

Presented as Paper 75-432 at the AIAA 11th Electric Propulsion Conference, New Orleans, La., March 19-21, 1975; submitted March 21, 1975; revision received Nov. 10, 1975. This research was supported by NASA.

Index categories: Electric and Advanced Space Propulsion; Plasma Dynamics and MHD.

*Professor of Electrical Engineering, Department of Electrical Engineering.

mal forces, which oppose compression, leads to the "overtaking of electron sheets"[1,2] and "multivalued flow"[1,2] only if the electrons are injected with a small density n_0. These phenomena cannot be treated within the frame of a) the zero-temperature approximation and b) the von Mises transformation,[9,10] on which the analysis is based. As will be shown, the theory is applicable to electron injection densities $N/2 \leq n_0 < \infty$.

III. Closed Form Solution

The nonlinear boundary-value problem defined in Eqs. (1-6) is solvable by a von Mises transformation.[9,10] Let a stream function $\psi = \psi(x,t)$ be introduced, which automatically satisfies Eq. (2), by the relations

$$\partial\psi/\partial t = nv/N \quad \partial\psi/\partial x = -n/N \tag{9}$$

The stream function is constant along the trajectories of the electron fluid since

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial x}\Big|_t \frac{dx}{dt} + \frac{\partial\psi}{\partial t}\Big|_x = -\frac{n}{N}v + \frac{nv}{N} = 0 \tag{10}$$

The Jacobian is

$$J\{[(x,t)/(\psi,t)]\} = \partial x/\partial\psi|_t \neq 0$$

for $n(x,t) < \infty$. For such solutions, it is permitted to introduce (ψ,t) as independent variables in place of (x,t):

$$\begin{gathered} x = x(\psi,t) \quad t = t \\ n(x,t) \to n(\psi,t) \quad E(x,t) \to E(\psi,t) \\ v(x,t) \to v(\psi,t) \end{gathered} \tag{11}$$

Since

$$v \equiv \frac{dx}{dt} = \frac{\partial x}{\partial\psi}\Big|_t \frac{d\psi}{dt} + \frac{\partial x}{\partial t}\Big|_\psi = \frac{\partial x}{\partial t}\Big|_\psi$$

that is

$$\partial v/\partial t|_\psi = \partial^2 x/\partial t^2|_\psi \tag{12}$$

and

$$\frac{dv}{dt} = \frac{\partial v}{\partial t}\Big|_x + v\frac{\partial v}{\partial x}\Big|_t = \frac{\partial v}{\partial\psi}\Big|_t \frac{d\psi}{dt} + \frac{\partial v}{\partial t}\Big|_\psi = \frac{\partial v}{\partial t}\Big|_\psi \tag{13}$$

it follows that

$$\partial v/\partial t + v(\partial v/\partial x) = \partial^2 x/\partial t^2|_\psi \tag{14}$$

Thus, by changing to the independent variables (ψ,t) and considering the interrelations in Eqs. (10, 11, and 14) the nonlinear Eqs. (1-3) are reduced to an inhomogeneous, damped, linear oscillator equation for $x = x(\psi,t)$:

$$\partial^2 x/\partial t^2 + \nu(\partial x/\partial t) + \omega^2 x = \nu V + \omega^2[f(t) - \psi] \tag{15}$$

where

$$f(t) \equiv -(Ne)^{-1}\int_0^t j(t')dt' \tag{16}$$

by the boundary condition in Eq. (4). The general solution of Eq. (15) is readily derived by means of Lagrange's method. From $x(\psi,t)$ one obtains $v(\psi,t)$ by partial differentiation $\partial/\partial t)_\psi$ of $x(\psi,t)$ [Eq. (12)], and $n(\psi,t)$ by partial differentiation $\partial/\partial\psi)_t$ of $x(\psi,t)$ [Eq. (9)], and $E(\psi,t)$ by subsequent integration of Eq. (3) and substitution of $x(\psi,t)$. The two integration constants $A(\psi)$ and $B(\psi)$ in $x(\psi,t)$ are determined by the boundary conditions in Eqs. (5) and (6). Thus, one finds the following analytical solutions of the nonlinear boundary-value problem in Eqs. (1-6):

$$E(\psi,t)/(m\omega^2/\sigma\Omega e) = \sigma\Omega\psi + \sigma\Omega x(\psi,t) - \Omega t H(t) \tag{17}$$

$$\begin{aligned} v(\psi,t)/\sigma^{-1} = {} & H(t) - [H(\tau) - N/n_0] \\ & \times \exp[-\nu(t-\tau)/2]\cos\Omega((t-\tau) - (\nu/2\Omega)\{[H(\tau) \\ & + N/n_0] - 2\sigma V\}\exp[-\nu(t-\tau)/2]\sin\Omega(t-\tau) \end{aligned} \tag{18}$$

$$\begin{aligned} n(\psi,t)/N = {} & H(\tau)[H(\tau) - \exp[-\nu(t-\tau)/2] \\ & \times\{[H(\tau) - N/n_0]\cos\Omega(t-\tau) + \{(\nu/2\Omega)[H(\tau) \\ & + N/n_0] - (\nu/\Omega)\sigma V\}\sin\Omega(t-\tau)\}]^{-1} \end{aligned} \tag{19}$$

where

$$\begin{aligned} \sigma\Omega x(\psi,t) = {} & \Omega t H(t) - \sigma\Omega\psi - (\nu\Omega/\omega^2)[H(t) - \sigma V] \\ & + \{(N/n_0) - [1 - 2(\nu/2\omega)^2]H(\tau) - 2(\nu/2\omega)^2\sigma V\} \\ & \exp[-\nu(t-\tau)/2]\sin\Omega(t-\tau) + \{(\nu\Omega/\omega)^2[H(\tau) \\ & - \sigma V]\}\exp[-\nu(t-\tau)/2]\cos\Omega(t-\tau) \end{aligned} \tag{20}$$

$$\tau H(\tau) = \sigma\psi \geq 0 \tag{21}$$

and

$$\Omega \equiv [\omega^2 - (\nu/2)^2]^{1/2} > 0 \tag{22}$$

is assumed to be real [see Eq. (8)].

Equations (17-19) represent a parametric solution of ψ, t where the parameter $\psi = \psi(x,t)$ is given for every pair of values of the independent variables x and t by Eq. (20). Since $\tau \equiv t$ for $x = 0$, Eqs. (17-20) give

$$E(x=0,t) = 0 \quad v(x=0,t) = v_0 \quad n(x=0,t) = n_0 H(t)$$

in accordance with Eqs. (4-6).

Physically, Eq. (20) gives the stream function $\psi = \psi(x,t)$, where $\psi(x,t) \geq 0$ by Eq. (21), for every point x of that region $x \geq 0$ which is occupied by electrons at time t. The limit, $\psi(x,t) = 0$, defines a function $\bar{x} = \bar{x}(t)$, which represents the moving position coordinate of the front of the electron gas. By Eqs. (17-19), the fields $E(\psi,t)$, $v(\psi,t)$, and $n(\psi,t)$ depend exclusively on $(t-\tau)$, i.e., on x by Eq. (20). The spatially periodic field configuration, which grows with the speed $d\bar{x}(t)/dt$ into the space $x \geq 0$, represents a (nonlinear) "neutralization wave." The appearance of discontinuous wave solutions is typical for hyperbolic problems.[9]

Equations (17-21) are based on a nonlinear transformation which exists if

$$J = \partial x(\psi,t)/\partial\psi \neq 0$$

or $n(\psi,t) < \infty$ in Eq. (19). Accordingly, the theory does not describe crossing of electron sheets,[1,2] i.e., it is restricted to electron injection densities n_0 above a minimum value

$$N/(2+\epsilon) < n_0 < \infty \quad \epsilon << 1 \quad \nu << 2\omega$$

by Eq. (19), which exhibits singularities for $n_0 < N/(2+\epsilon)$. For the purpose of neutralization, electron injection densities $n_0 < N/2$ are hardly of any technical interest.

IV. Dissipative Neutralization

Equations (17-21) describe a nonlinear electron neutralization wave which is spatially damped as a result of the irreversible friction between the electron and ion streams. The position $\bar{x}(t)$ of the neutralization front is obtained from Eq. (20) as the limit $\bar{x}(t) = x(\psi \to 0, t)$

$$\sigma\Omega\bar{x}(t) = \Omega t H(t) - (\nu\Omega/\omega^2)[H(t) - \sigma V] + [\{(N/n_0) - [1 - 2(\nu/2\omega)^2] - 2(\nu/2\omega)^2\sigma V\}\sin\Omega t + \{(\nu\Omega/\omega)^2[1 - \sigma V]\}\cos\Omega t]\exp(-\nu t/2) \tag{23}$$

by Eq. (21). It follows for the speed of the neutralization front

$$v(\psi = 0, t)/\sigma^{-1} = H(t) - \{(1 - N/n_0)\cos\Omega t + (\nu/2\Omega)[1 + (N/n_0) - 2\sigma V]\sin\Omega t\}\exp(-\nu t/2) \tag{24}$$

by Eq. (18). In the region ahead of the neutralization front, $x > \bar{x}(t)$, it is $\tau < 0$ and $\psi = 0$ by Eq. (21). Accordingly,

$$n(x,t) = 0 \quad v(x,t) = 0 \quad \text{for } x > \bar{x}(t) > 0$$

whereas $E(x,t) \neq 0$ for $x > \bar{x}(t)$ by Eq. (17). Equation (17) yields

$$\partial E/\partial x = m\omega^2 e \text{ and } \partial E/\partial t = -m\omega^2/\sigma e \text{ for } \psi = 0$$

in accordance with Eqs. (2) and (3) for $n = 0$.

For the graphical representation of the wave fields $n(x,t)$, $E(x,t)$, and $v(x,t)$, let nondimensional variables and fields be introduced by the substitutions

$$x/x_0 = \hat{x} \quad t/t_0 = \hat{t} \quad \psi/x_0 = \hat{\psi} \tag{25a}$$

$$E/E_0 = \hat{E} \quad v/V_0 = \hat{v} \quad V/V_0 = \hat{V} \quad n/N_0 = \hat{n} \tag{25b}$$

where

$$x_0 \equiv 1/\sigma\Omega \quad t_0 \equiv 1/\Omega \quad E_0 \equiv m\omega^2/\sigma\Omega e \tag{26a}$$

$$V_0 \equiv 1/\sigma \quad N_0 \equiv N \quad \alpha \equiv N/n_0 \tag{26b}$$

into the Eqs. (17-21). The $(e-i)$ momentum transfer frequency ν is calculated for ions of the mass M of mercury by means of Eq. (8).

In Fig. 2, the nondimensional fields $\hat{n}(\hat{x},\hat{t})$, $\hat{E}(\hat{x},\hat{t})$, and $\hat{v}(\hat{x},\hat{t})$ [Eqs. (17-19)] of the neutralization wave are shown vs $0 \leq \hat{x} \leq \bar{x}(t)/x_0$ at time $t = 10\pi$, with $\alpha = 0$, 1.1, and $\hat{V} = 10^2\alpha$ as parameters, and

$$\nu/2\omega = (m/M)^{1/3}/4\pi = 1.107 \times 10^{-3}$$

for mercury ions. As a result of the periodic over and under "neutralization" (Fig. 2a), the self-consistent electric field $\hat{E}(\hat{x}, \hat{t})$ periodically changes its direction (Fig. 2b). It is $\bar{x}(t = 10\pi)/x_0 \cong 20\pi$ by Eq. (23). For other times $\hat{t} = \hat{t}$ the fields of the neutralization wave exist only up to the point $\hat{x} = \bar{x}(\hat{t})$, e.g., the broken lines in Fig. 2 represent the neutralization front at time $\hat{t} = 2\pi$.

The momentum transfer between the electron stream and the ion beam results in an amplitude asymmetry which is clearly visible in the case $\alpha = 1.1$ of Fig. 2 and damping of the standing wave amplitudes as shown in Fig. 2. Equation (20) shows that, for large values of $(t - \tau)$, $x(\psi, t)$ approaches the limit (dimensional)

$$\sigma x = t - \tau - \nu\omega^{-2}(1 - \sigma V) \quad t - \tau >> 2/\nu \tag{27}$$

Hence

$$n(x,t) \to N \quad v(x,t) \to \sigma^{-1} \quad \text{for } \Lambda << x \leq \bar{x}(t) \tag{28}$$

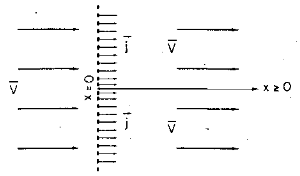

Fig. 1 Schematic representation of infinite ion beam (V) with electron current injection j from grid plane at $x = 0$.

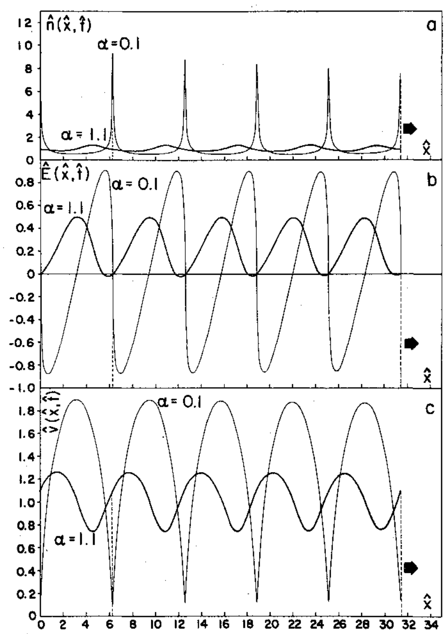

Fig. 2 Nondimensional a) electron density field $\hat{n}(\hat{x},\hat{t})$, b) electric field $\hat{E}(\hat{x},\hat{t})$, and c) velocity field $\hat{v}(\hat{x},\hat{t})$ of neutralization wave vs nondimensional $\hat{x}$ for $\hat{t} = 10\pi$ and $\alpha = 0.1$; 1.1.

$$E(x,t) \to -(\nu m/\sigma e)(1 - \sigma V) \quad \text{for } \Lambda << x \leq \bar{x}(t) \tag{29}$$

where

$$\Lambda = 2/\sigma\nu \equiv (2/\nu)|i_0|/Ne \tag{30}$$

by Eqs. (17-19) and (27). Equation (28) indicates that the electron density n equals the ion density N for $\Lambda << x \leq \bar{x}(t)$, and the electron stream velocity is

$$v(x,t) = (n_0/N)v_0 \quad \Lambda << x \leq \bar{x}(t) \tag{31}$$

This means that a complete neutralization ($n = N$) is reached about three relaxation lengths Λ downstream of the emitter at $x = 0$. The electron current density $i_0 = -en_0v_0$ is conserved before and after the neutralization since

$$n(x,t)v(x,t) = N\cdot(n_0/N)v_0 = n_0v_0 \quad \Lambda << x \leq \bar{x}(t) \tag{32}$$

by Eq. (28). In the neutralized region, $3\Lambda \lesssim x \leq \bar{x}(t)$, of the ion beam, the electric and inter-component friction forces are in balance

$$-eE - \nu m[(n_0/N)v_0 - V] = 0 \quad \Lambda << x \leq \bar{x}(t) \tag{33}$$

by Eq. (29). It is remarkable that the neutralization is brought about by irreversible momentum exchange between the electron and ion streams. The relaxation length for neutralization decreases as the electron-ion interaction increase, and $\Lambda \to \infty$ for $\nu \to 0$ by Eq. (30). Figure 2 shows the neutralization wave in the initial stage before any significant neutralization has occurred, so that $\bar{x}(\hat{t} = 10\pi) << \Lambda$.

For neutralization experiments, it is noted that the relaxation length Λ decreases with increasing ion density N and decreasing electron current $|i_0|$, as well as increasing interaction frequency $\nu(N)$. In Fig. 3, the relaxation length Λ

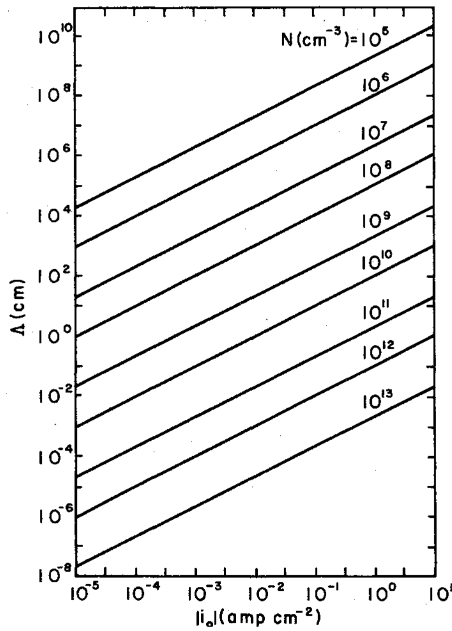

Fig. 3 Relaxation length Λ in dependence of $|i_0|$ and N.

for neutralization is plotted in dependence of the electron injection current $|i_0|$ [amp-cm^{-2}] with the ion density N[cm^{-3}] as a parameter for mercury ions ($M=200.59$ amu) and the interaction frequency in Eq. (8).

The reference values x_0, t_0, E_0, V_0, and N_0 defined in Eq. (25) determine the characteristic scales and actual magnitude of the fields of the neutralization wave. As a numerical example, consider a mercury ion beam and an electron injection current for which

$$N=10^9\text{cm}^{-3} \quad |i_0| \hat{=} 10^{-3}\text{amp-cm}^{-2}$$

$$\beta=\nu/2\omega=1.107\times10^{-3}$$

Hence

$$x_0=3.691\times10^4(1-\beta^2)^{-1/2}|i_0|N^{-3/2}=3.502\times10^{-3}\text{cm}$$

$$t_0=1.773\times10^{-5}(1-\beta^2)^{-1/2}N^{-1/2}=5.607\times10^{-10}\text{sec}$$

$$E_0=2.227\times10^{-4}(1-\beta^2)^{1/2}|i_0|N^{-1/2}$$

$$=2.113\times10^{-2}\text{cgsu}=6.339\times10^0\text{V-cm}^{-1}$$

$$V_0=2.082\times10^9|i_0|N^{-1}=6.245\times10^6\text{cm-sec}^{-1}$$

$$N_0=N=10^9\text{cm}^{-3}$$

and

$$\Lambda=3.691\times10^4\beta^{-1}|i_0|N^{-3/2}=3.163\times10^0\text{cm}$$

For the above numerical values of N, $|i_0|$, and β, the "wavelength" of the neutralization wave in Fig. 2 is $\Delta x \cong 6x_0=2.101\times10^{-2}$cm. The electron density in Fig. 2a fluctuates by $+\Delta n\cong0.9N_0=9\times10^8\text{cm}^{-3}$ for $N/n_0=0.1$ and by $+\Delta n\cong0.2N_0=2\times10^8\text{cm}^{-3}$ for $N/n_0=1.1$ in the unneutralized ion beam. The electric field in Fig. 2b fluctuates by $\pm\Delta E=0.9E_0=5.705$ volt cm^{-1} for $N/n_0=0.1$ and by $\pm\Delta E\cong0.5E_0=3.170$ V-cm^{-1} for $N/n_0=1.1$. The velocity field in Fig. 2c fluctuates by $\pm\Delta v\cong0.9V_0=5.621\times10^6$ cm-sec^{-1} for $N/n_0=0.1$ and by $\pm\Delta v=0.3V_0=1.874\times10^6$cm-sec^{-1} for $N/n_0=1.1$. A complete neutralization should be achieved after a length of about $3\Lambda=9.489$ cm. It is noted that the magnitudes of the relaxation length Λ and of the fluctuations Δn, ΔE, Δv change considerably as i_0 and N are varied, since $\Lambda\propto|i_0|\,N^{-3/2}$, $N_0=N$, $E_0\propto|i_0|\,N^{-1/2}$ and $V_0\propto|i_0|\,N^{-1}$ while $\beta=\nu/2\omega=(m/M)^{1/3}/4\pi$ is fixed.

V. Conclusions

The neutralization of ion beams occurs through a nonlinear, discontinuous electron wave, which is spatially attenuated as a result of dissipation, with a neutralization length Λ [Eq. (30)]. In the absence of irreversible momentum transfer between the electron and ion components, neutralization is not possible ($\Lambda=\infty$ for $\nu=0$).

If the electrons are injected with a density n_0, which is equal to the ion density N, then the neutralization occurs through a discontinuous electron wave of the form of a propagating step impulse of height $n_0=N$. This special case represents the optimum situation for neutralization experiments.

The neutralization wave theory presented explains the neutralization observed in ion beams of ion propulsion systems[11,12] Equation (30) gives a neutralization length Λ of the same order of magnitude as the observed neutralization length.[11-12]

Measurements on the temporal and spatial development of the neutralization and the "fine-structure" of the incompletely neutralized ion beam have evidently not been reported. Only very recently, phenomena such as low-frequency fluctuations in ion beams have been studied experimentally.[13]

References

[1] Buneman, O., *Physical Review,* Vol. 115, 1959, pp. 503-517.

[2] Dawson, J.M., *Physical Review,* Vol. 113, 1959, pp. 383-387.

[3] Stuhlinger, E., *Ion Propulsion for Space Flight,* McGraw Hill, New York, 1964, pp. 226.

[4] Au, G.F., *Electric Propulsion of Space Vehicles,* Verlag G. Brown, Karlsruhe, W. Germany, 1968, p. 439.

[5] Kaufman, H.R., NASA TN D-261, 1960.

[6] Pearlstein, L.D., Rosenbluth, M.N., and Stuart, G.W., "Neutralization of Ion Beams," *AIAA Progress in Astronautics and Aeronautics: Electic Propulsion Development,* Vol. 9, edited by Ernst Stuhlinger, Academic Press, New York, 1963, pp. 379-406.

[7] Vlasov, A.A., *Many Particle Theory and its Application to Plasma,* Gordon & Breach, New York, 1961, p. 327.

[8] Montgomery, D. and Gorman, D., *Physics of Fluids,* Vol. 5, 1962, pp. 947-949.

[9] Courant, R. and Friedrichs, K.O., *Supersonic Flow and Shock Waves,* Wiley Interscience, New York, 1948, p. 248.

[10] Kalman, G., *Ann. Phys.*, Vol. 10, 1960, pp. 1-28.

[11] Kaufman, H.R., *Advance Electronics and Electron Physics,* Vol. 36, 1974, pp. 265-348.

[12] Kerslake, N.R., Byers, D.C., and Staggs, J.F., "SERT II: Mission and Experiments," *Journal of Spacecraft and Rockets,* Vol. 7, Jan. 1970, pp. 4-6.

[13] Serafini, J.S. and Terdan, F.F., NASA TM X-71421, 1973.